$\underline{\underline{Th^m}}$: State and prove Gauss Divergence theorem. (1)

Statement: Suppose V is the volume bounded by a closed piecewise smooth surface S. Suppose $\vec{F}$ is a vector point function which is continuous and has continuous first order partial derivatives in V, then

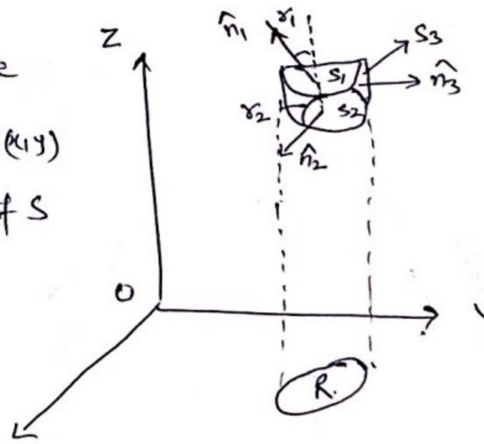
$$\iiint_V \nabla\cdot\vec{F}\, dv = \iint_S \vec{F}\cdot\hat{n}\, ds$$

where $\hat{n}$ is the unit outwards drawn normal vector to the surface S.

Proof: We shall first prove the th^m for a special volume V which is bounded by a piecewise smooth closed surface S and has the property that any straight line ||al to any one of the co-ordinate axes and intersecting V has only one segment (or a single point) in common with V.

If R is the orthogonal projection of S on xy-plane, then V can be represented in the form $f(x,y) \le z \le g(x,y)$, where (x,y) varies in R.

Clearly $z = g(x,y)$ represents the upper portion S_1 of S and $z = f(x,y)$ represents the lower portion S_2 of S and there may be a remaining vertical portion S_3 of S.



let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

then
$$\iiint_V \frac{\partial F_3}{\partial z}\, dv = \iiint_V \frac{\partial F_3}{\partial z}\, dx\,dy\,dz = \iint_R \left[\int_{z=f(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z}\, dz\right] dx\,dy$$

$$= \iint_R \left[F_3(x,y,z)\right]_{z=f(x,y)}^{g(x,y)} dx\,dy$$

$$= \iint_R F_3[x,y,g(x,y)]\,dx\,dy - \iint_R F_3[x,y,f(x,y)]\,dx\,dy \quad \text{---} (1)$$

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Now for the vertical portion S_3 of S, the normal $\hat{n}_3$ to S_3 makes a right angle with $\hat{k}$. Therefore

$$\iint_{S_3} F_3\, \hat{k}\cdot\hat{n}_3\, dS_3 = 0 \qquad (\because \hat{k}\cdot\hat{n}_3 = 0) \quad \text{—— } (2)$$

For the upper portion S_1 of S the normal $\hat{n}_1$ to S_1 makes an acute angle γ_1 with $\hat{k}$. Therefore

$$\hat{k}\cdot\hat{n}_1\, dS_1 = \cos\gamma_1\, dS_1 = dx\,dy$$

Hence $$\iint_{S_1} F_3\, \hat{k}\cdot\hat{n}_1\, dS_1 = \iint_R F_3[x, y, g(x,y)]\,dx\,dy \quad \text{—— } (3)$$

For the lower portion S_2 of S, the normal $\hat{n}_2$ to S_2 makes an obtuse angle γ_2 with $\hat{k}$. Therefore

$$\hat{k}\cdot\hat{n}_2\, dS_2 = \cos\gamma_2\, dS_2 = -dx\,dy$$

$$\left[\because dx\,dy = dS_2 \cos(\pi - \gamma_2) = -dS_2\cos\gamma_2, \right.$$
$$\text{But } \hat{k}\cdot\hat{n}_2 = \cos\gamma_2 \Rightarrow -\hat{k}\cdot\hat{n}_2\, dS_2 = dx\,dy$$

Hence $$\iint_{S_2} F_3\, \hat{k}\cdot\hat{n}_2\, dS_2 = -\iint_R F_3[x, y, f(x,y)]\,dx\,dy \quad \text{—— } (4)$$

Adding (2), (3) and (4), we get

$$\iint_{S_3} F_3\, \hat{k}\cdot\hat{n}_3\, dS_3 + \iint_{S_1} F_3\, \hat{k}\cdot\hat{n}_1\, dS_1 + \iint_{S_2} F_3\, \hat{k}\cdot\hat{n}_2\, dS_2$$
$$= 0 + \iint_R F_3[x, y, g(x,y)]\,dx\,dy - \iint_R F_3[x, y, f(x,y)]\,dx\,dy$$

Therefore from (1), we have

$$\iiint_V \frac{\partial F_3}{\partial z}\, dv = \iint_{S_1} F_3\, \hat{k}\cdot\hat{n}_1\, dS_1 + \iint_{S_2} F_3\, \hat{k}\cdot\hat{n}_2\, dS_2 + \iint_{S_3} F_3\, \hat{k}\cdot\hat{n}_3\, dS_3$$
$$= \iint_S F_3\, \hat{k}\cdot\hat{n}\, dS \quad \text{—— } (5)$$

Similarly by projecting S on the other co-ordinate planes, we get

$$\iiint_V \frac{\partial F_1}{\partial x}\, dv = \iint_S F_1\, \hat{i}\cdot\hat{n}\, dS \quad \text{—— } (6)$$

and $$\iiint_V \frac{\partial F_2}{\partial y}\, dv = \iint_S F_2\, \hat{j}\cdot\hat{n}\, dS \quad \text{—— } (7)$$

Adding ⑤, ⑥ and ⑦, we get

$$\iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dv = \iint_S (f_1\hat{i} + f_2\hat{j} + f_3\hat{k}) \cdot \hat{n}\, ds$$

$$\Rightarrow \iiint_V \nabla \cdot \vec{F}\, dv = \iint_S \vec{F} \cdot \hat{n}\, ds$$

The proof of this thm can now be extended to a region V which can be subdivided into finitely many special regions of the above type of drawing auxillary surfaces. In this case, we apply the thm to each sub region and then add the results. The sum of volume integrals over parts of V will be equal to the volume integral over V. The surface integrals over auxillary surface cancel in pairs, whereas the sum of remaining surface integrals is equal to the surface integrals over the entire boundary of V.

<u>Note</u>: With the help of "Gauss Divergence thm" we can express a volume integral as a surface integral or a surface integral as volume integral.

<u>Thm</u>: Let ϕ and ψ be scalar point functions which together with their derivatives in any direction are continuous within region V bounded by a closed surface S

Then $\iiint_V (\nabla\phi \cdot \nabla\psi + \phi\nabla^2\psi)\, dv = \iint_S (\phi\nabla\psi) \cdot \hat{n}\, ds$ (Green's I identity)

Further $\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dv = \iint_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \hat{n}\, ds$

<u>Proof</u>: By Gauss divergence thm, we have

$$\iiint_V \nabla \cdot \vec{F}\, dv = \iint_S \vec{F} \cdot \hat{n}\, ds$$

Taking $\vec{F} = \phi \nabla \psi$, we have

$$\iiint_V \nabla \cdot (\phi \nabla \psi)\, dv = \iint_S (\phi \nabla \cdot \psi) \cdot \hat{n}\, ds$$

$$\Rightarrow \iiint_V [\nabla \phi \cdot \nabla \psi + \phi (\nabla \cdot \nabla \psi)]\, dv = \iint_S (\phi \nabla \psi) \cdot \hat{n}\, ds$$

$\left[\because \text{div}(u\vec{v}) = \nabla u \cdot \vec{v} + u \nabla \cdot \vec{v} \right.$

$$\Rightarrow \iiint_V (\nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi)\, dv = \iint_S (\phi \nabla \psi) \cdot \hat{n}\, ds \quad \text{--- (1)}$$

which proves Green's I identity.

Interchanging ϕ and ψ in (1), we get

$$\iiint_V (\nabla \psi \cdot \nabla \phi + \psi \nabla^2 \phi)\, dv = \iint_S (\psi \nabla \phi) \cdot \hat{n}\, ds \quad \text{--- (2)}$$

(1) - (2), we have

$$\iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dv = \iint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \hat{n}\, ds$$

which proves Green's II identity

Cor: If ϕ and ψ are harmonic functions i.e. $\nabla^2 \phi = \nabla^2 \psi = 0$ then by Green's II identity, we have

$$\iint (\phi \nabla \psi - \psi \nabla \phi) \cdot \hat{n}\, ds = 0.$$

Th^m: If ϕ and $\vec{F}$ are point functions, then

(i) $\iint_S \hat{n} \times \vec{F}\, ds = \iiint_V \nabla \times \vec{F}\, dv$

(ii) $\iint_S \phi\, \hat{n}\, ds = \iiint_V \nabla \phi\, dv$, where S is closed surface and V is volume enclosed by S.

Proof: By Gauss's Divergence th^m, we have

$$\iint_S \vec{f} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{f}\, dv \quad \text{--- (1)}$$

(i) Taking $\vec{f} = \vec{a} \times \vec{F}$, where $\vec{a}$ is arbitrary constant vector and $\vec{F}$ is a vector pt. function, then from (1), we get

$$\iint_S (\vec{a} \times \vec{F}) \cdot \hat{n}\, ds = \iiint_V \nabla \cdot (\vec{a} \times \vec{F})\, dv$$

$$\Rightarrow \iint_S \vec{a} \cdot (\vec{F} \times \hat{n})\, ds = \iiint_V [\vec{F} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{F})]\, dv$$

$$\Rightarrow \vec{a}\cdot\iint_S (\vec{F}\times\hat{n})\,ds = -\iiint_V \vec{a}\cdot(\nabla\times\vec{F})\,dv \qquad [\because \nabla\times\vec{a}=\vec{0}]$$

$$\Rightarrow -\vec{a}\cdot\iint_S \hat{n}\times\vec{F}\,ds = -\vec{a}\cdot\iiint_V \nabla\times\vec{F}\,dv$$

$$\Rightarrow \vec{a}\cdot\left[\iint_S \hat{n}\times\vec{F}\,ds - \iiint_V \nabla\times\vec{F}\,dv\right]=0$$

$$\Rightarrow \iint_S \hat{n}\times\vec{F}\,ds - \iiint_V \nabla\times\vec{F}\,dv = \vec{0} \qquad \left[\because \vec{a} \text{ is arbitrary constant vector}\right]$$

$$\Rightarrow \iint_S \hat{n}\times\vec{F}\,ds = \iiint_V \nabla\times\vec{F}\,dv$$

i.e. $\iint_S \vec{F}\times\hat{n}\,ds = -\iiint_V \text{Curl}\,\vec{F}\cdot dv$

(ii) Taking $\vec{f}=\phi\,\vec{a}$, where $\vec{a}$ is arbitrary constant vector and ϕ is a scalar pt function, from ① we get

$$\iint_S \phi\,\vec{a}\cdot\hat{n}\,ds = \iiint_V \text{div}(\phi\,\vec{a})\,dv = \iiint_V (\nabla\phi\cdot\vec{a}+\phi\,\nabla\cdot\vec{a})\,dv$$

$$= \iiint_V \vec{a}\cdot\nabla\phi\,dv \qquad [\because \nabla\cdot\vec{a}=0]$$

$$\Rightarrow \vec{a}\cdot\iint_S \phi\hat{n}\,ds = \vec{a}\cdot\iiint_V \nabla\phi\,dv \Rightarrow \vec{a}\cdot\left[\iint_S \phi\hat{n}\,ds - \iiint_V \nabla\phi\,dv\right]=0$$

$$\Rightarrow \iint_S \phi\hat{n}\,ds - \iiint_V \nabla\phi\,dv = \vec{0} \qquad [\because \vec{a} \text{ is arbitrary constant vector}]$$

$$\Rightarrow \iint_S \phi\hat{n}\,ds = \iiint_V \nabla\phi\,dv$$

Q: If S is a closed surface enclosing volume V, then show that $\iint_S r^5\,\hat{n}\,ds = \iiint_V 5r^3\,\vec{r}\,dv$.

Sol: We know that $\iint_S \phi\,\hat{n}\,ds = \iiint_V \nabla\phi\,dv$ —①

Taking $\phi = r^5$ in ①, we have

$$\iint_S r^5\,\hat{n}\,ds = \iiint_V (\nabla r^5)\,dv \qquad [\because \nabla r^n = n r^{n-2}\vec{r}]$$

$$= \iiint_V 5r^3\,\vec{r}\,dv$$

Q: If S is closed surface enclosing volume V, then show that $\iint_S (\nabla\phi\times\nabla\psi)\cdot d\vec{s} = 0$

Sol^n: $\iint_S (\nabla\phi \times \nabla\psi)\cdot d\vec{s} = \iint_S (\nabla\phi\times\nabla\psi)\cdot\hat{n}\,ds$

$= \iiint_V \nabla\cdot(\nabla\phi\times\nabla\psi)\,dv$ [By divergence-th^m]

$= \iiint_V [\{\nabla\times(\nabla\phi)\}\cdot\nabla\psi - \nabla\phi\cdot\{\nabla\times(\nabla\psi)\}]\,dv$

$[\because \nabla\cdot(\bar{f}\times\bar{g}) = (\nabla\times\bar{f})\cdot\bar{g} - \bar{f}\cdot(\nabla\times\bar{g})]$

$= 0$ $\quad [\because \nabla\times(\nabla\phi) = 0 = \nabla\times(\nabla\psi)]$

Q: Prove that $\iint_S \bar{r}\times\hat{n}\,ds = \vec{0}$, for any closed surface S.

Sol^n: $\iint_S \bar{r}\times\hat{n}\,ds = -\iint_S \hat{n}\times\bar{r}\,ds = -\iiint_V (\nabla\times\bar{r})\,dv = \vec{0}$

$\left[\because \iint_S \hat{n}\times\bar{F}\,ds = \iiint_V \nabla\times\bar{F}\,dv \text{ and } \nabla\times\bar{r} = \vec{0}\right]$

Q: Show that $\iint_S \left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\phi}{\partial n}\right]ds = -\iiint_V \frac{1}{r}\nabla^2\phi\,dv$

where S is closed surface enclosing a region v and $r = \sqrt{x^2+y^2+z^2}$

Sol^n: $\iint_S \left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\phi}{\partial n}\right]ds = \iint_S \left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\hat{n} - \frac{1}{r}\frac{\partial\phi}{\partial n}\hat{n}\right]\cdot\hat{n}\,ds$

$= \iint_S \left[\phi\nabla\left(\frac{1}{r}\right) - \frac{1}{r}\nabla\phi\right]\cdot\hat{n}\,ds$

$= \iiint_V \nabla\cdot\left[\phi\nabla\left(\frac{1}{r}\right) - \frac{1}{r}\nabla\phi\right]dv$ [by divergence th^m]

$= \iiint_V \left[\underline{\nabla\phi\cdot\nabla\left(\frac{1}{r}\right)} + \phi\nabla\cdot\nabla\left(\frac{1}{r}\right) - \underline{\nabla\left(\frac{1}{r}\right)\cdot\nabla\phi} - \frac{1}{r}\nabla\cdot\nabla\phi\right]dv$

$[\because \nabla\cdot(u\bar{v}) = \nabla u\cdot\bar{v} + u(\nabla\cdot\bar{v})]$

$= \iiint_V \left[\phi\nabla^2\left(\frac{1}{r}\right) - \frac{1}{r}\nabla^2\phi\right]dv$

$= -\iiint_V \frac{1}{r}\nabla^2\phi\,dv$ $\quad \left[\because \nabla^2\left(\frac{1}{r}\right) = 0\right]$

Q: For a closed surface S, and point functions $\bar{F}, v$, prove that integrals $\iint_S \hat{n}\cdot \text{Curl}\,\bar{F}\,ds$ and $\iint_S \hat{n}\times\nabla v\,ds$ vanish identically.

Sol^n: By Gauss divergence th^m, we have

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\bar{F}\,dv \quad \text{—} (1)$$

Replacing $\bar{F}$ by $\text{Curl}\,\bar{F}$ in (1), we get

$$\iint_S \text{Curl}\,\bar{F}\cdot\hat{n}\,ds = \iiint_V \text{div}(\text{Curl}\,\bar{F})\,dv = 0$$

$$[\because \text{div}(\text{Curl}\,\bar{F}) = 0].$$

Also we have by th^m on page (4)

$$\iint_S \hat{n}\times\bar{F}\,ds = \iiint_V \nabla\times\vec{F}\,dv \quad \text{—} (2)$$

Replace $\bar{F}$ by ∇v in (2), we have

$$\iint_S \hat{n}\times\bar{F}\,ds = \iint_S \hat{n}\times\nabla v\,ds = \iiint_V \nabla\times\nabla v = \bar{0}$$

$$\Rightarrow \iint_S \hat{n}\times\nabla v\,ds = \bar{0} \qquad [\because \nabla\times\nabla v = \text{Curl}(\text{grad}\,v) = \bar{0}]$$

Q: If $\hat{n}$ is unit outward drawn normal to any closed surface S, show that $\iiint_V \text{div}\,\hat{n}\,dv = S$

Sol^n: By Gauss divergence th^m $\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\bar{F}\,dv.$

Put $\bar{F} = \hat{n}$ we get

$$\iiint_V \text{div}\,\hat{n}\,dv = \iint_S \hat{n}\cdot\hat{n}\,ds = \iint_S ds = S$$

Q: If S is closed surface enclosing volume V, then prove that $\iiint_V (\nabla\phi)\cdot\bar{A}\,dv = \iint_S \phi\bar{A}\cdot\hat{n}\,ds - \iiint_V \phi(\nabla\cdot\bar{A})\,dv$

Sol^n By Gauss divergence th^m we have

$$\iiint_V \nabla\cdot(\phi\bar{A})\,dv = \iint_S (\phi\bar{A})\cdot\hat{n}\,ds \qquad \left[\because \iiint_V \nabla\cdot\bar{F}\,dv = \iint_S \bar{F}\cdot\hat{n}\,ds\right]$$

$$\Rightarrow \iiint_V [(\nabla\phi)\cdot\bar{A} + \phi(\nabla\cdot\bar{A})]\,dv = \iint_S (\phi\bar{A})\cdot\hat{n}\,ds$$

$$\Rightarrow \iiint_V \nabla\phi\cdot\vec{A}\,dv + \iiint_V \phi(\nabla\cdot\bar{A})\,dv = \iint_S (\phi\bar{A})\cdot\hat{n}\,ds$$

$$\Rightarrow \iiint_V \nabla\phi\cdot\vec{A}\,dv = \iint_S \phi\bar{A}\cdot\hat{n}\,ds - \iiint_V \phi(\nabla\cdot\bar{A})\,dv$$

Q: Show that $\iiint_V \bar{v}\cdot\nabla p\,dv = \iint_S p\bar{v}\cdot\hat{n}\,ds - \iiint_V p\,\mathrm{div}\,\vec{v}\,dv$.

and $\iiint_V \bar{v}\cdot \mathrm{Curl}\,\vec{u}\,dv = \iint_S \bar{U}\times\bar{V}\cdot\hat{n}\,ds + \iiint_V \bar{U}\cdot\mathrm{Curl}\,\bar{v}\,dv$

Sol: By Guass thm $\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \mathrm{div}\,\bar{F}\,dv$ — (1)

Put $\bar{F} = p\bar{v}$ in (1), we get

$$\iint_S p\bar{v}\cdot\hat{n}\,ds = \iiint_V \mathrm{div}(p\bar{v})\,dv = \iiint_V [(\nabla p)\cdot\bar{v} + p\,\mathrm{div}\,\bar{v}]\,dv$$

$$\Rightarrow \iint_S p\bar{v}\cdot\hat{n}\,ds = \iiint_V (\nabla p)\cdot\bar{v}\,dv + \iiint_V p\,\mathrm{div}\,\bar{v}\,dv$$

$$\Rightarrow \iiint_V \bar{v}\cdot(\nabla p)\,dv = \iint_S p\bar{v}\cdot\hat{n}\,ds - \iiint_V p\,\mathrm{div}\,\bar{v}\,dv$$

For 2nd part ~~part~~ ~~We know that~~ put $\bar{F} = \bar{U}\times\bar{V}$ in (1)

we get
$$\iint_S (\bar{U}\times\bar{V})\cdot\hat{n}\,ds = \iiint_V \mathrm{div}(\bar{U}\times\bar{V})\,dv$$

$$\Rightarrow \iint_S (\bar{U}\times\bar{V})\cdot\hat{n}\,ds = \iiint_V [\bar{V}\cdot\mathrm{Curl}\,\bar{U} - \bar{U}\cdot\mathrm{Curl}\,\bar{V}]\,dv$$

$$= \iiint_V \bar{V}\cdot\mathrm{Curl}\,\vec{U}\,dv - \iiint_V \bar{U}\cdot\mathrm{Curl}\,\bar{V}\,dv$$

$$\Rightarrow \iiint_V \bar{V}\cdot\mathrm{Curl}\,\bar{U}\,dv = \iint_S (\bar{U}\times\bar{V})\cdot\hat{n}\,ds + \iiint_V \bar{U}\cdot\mathrm{Curl}\,\bar{V}\,dv$$

Q: If $\bar{F} = \nabla\phi$ and $\nabla^2\phi = 0$, show that for a closed surface S, $\iiint_V \vec{F}^2\,dv = \iint_S \phi\bar{F}\cdot\hat{n}\,ds$

Sol: By divergence thm

$$\iint_S (\phi\bar{F})\cdot\hat{n}\,ds = \iiint_V \mathrm{div}(\phi\bar{F})\,dv = \iiint_V [\nabla\phi\cdot\bar{F} + \phi(\nabla\cdot\bar{F})]\,dv$$

$$= \iiint_V [\bar{F}\cdot\bar{F} + \phi(\nabla\cdot\nabla\phi)]\,dv = \iiint_V (F^2 + \phi\nabla^2\phi)\,dv$$

$$= \iiint F^2\,dv \qquad [\because \nabla^2\phi = 0]$$

Q: If $\bar{F} = \nabla V$ and $\nabla^2 V = -4\pi f$, Show that

$$\iint_S \bar{F}\cdot\hat{n}\, ds = -4\pi \iiint_V f\, dv.$$

Solⁿ By divergence thᵐ $\iint_S \bar{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\bar{F}\, dv$

$$\Rightarrow \iint_S \bar{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot(\nabla V)\, dv \qquad [\because \bar{F} = \nabla V]$$

$$= \iiint_V \nabla^2 V\, dv = \iiint_V (-4\pi f)\, dv$$

$$= -4\pi \iiint_V f v$$

Q: If $c f \frac{\partial v}{\partial t} = \operatorname{div}(k\nabla v)$, and v is zero over closed surface S, show that $\iiint_V c f v \frac{\partial v}{\partial t}\, dv = -\iiint_V k(\nabla v)^2 dv.$

Solⁿ: By Green's Ist identity

$$\iiint_V (\nabla U\cdot\nabla V)\, dv = \iint_S U\hat{n}\cdot\nabla V\, ds - \iiint_V U\nabla^2 V\, dv$$

Replacing U by V, we get

$$\iiint_V (\nabla V\cdot\nabla V)\, dv = \iint_S V\hat{n}\cdot\nabla V\, ds - \iiint_V V\nabla^2 V\, dv$$

But V is zero over a closed surface.

$$\therefore \iiint_V (\nabla V)^2 dv = -\iiint_V V\nabla^2 V\, dv \Rightarrow \iiint_V k(\nabla v)^2 dv + \iiint_V v k \nabla^2 v = 0$$

$$\Rightarrow \iiint_V k(\nabla v)^2 dv + \iiint_V v \operatorname{div}(k\nabla v)\, dv = 0$$

$$\Rightarrow \iiint k(\nabla v)^2 dv + \iiint v\, c f \frac{\partial v}{\partial t}\, dv = 0 \qquad \left[\because \operatorname{div}(k\nabla v) = c f \frac{\partial v}{\partial t}\right]$$

$$\Rightarrow \iiint c f v \frac{\partial v}{\partial t}\, dv = -\iiint k(\nabla v)^2 dv$$

Q: Expanding $\operatorname{div}(\nabla U\times\bar{V})$ and using divergence thᵐ, show that $-\iint \nabla U\times\bar{V}\cdot\hat{n}\, ds = \iiint \nabla U\cdot\nabla\times\bar{V}\, dv$

Solⁿ We know that $\operatorname{div}(\bar{f}\times\bar{g}) = \bar{g}\cdot\operatorname{curl}\bar{f} - \bar{f}\cdot\operatorname{curl}\bar{g}$

$$\therefore \operatorname{div}(\nabla U\times\bar{V}) = \bar{V}\cdot\operatorname{curl}(\nabla U) - \nabla U\cdot\operatorname{curl}\bar{V} \quad \text{—(1)}$$

By divergence th^m $\iint \bar{F}\cdot\hat{n}\,ds = \iiint \operatorname{div}\bar{F}\,dv$

Replace $\bar{F}$ by $\nabla U\times\bar{V}$, we have

$$\iint(\nabla U\times\bar{V})\cdot\hat{n}\,ds = \iiint \operatorname{div}(\nabla U\times\bar{V})\,dv$$

$$= \iiint[\bar{V}\cdot \operatorname{Curl}\nabla U - \nabla U\cdot\operatorname{Curl}\bar{V}]\,dv \quad [\text{by } ①]$$

$$= -\iiint \nabla U\cdot \operatorname{Curl}\bar{V}\,dv \quad [\because \operatorname{Curl}\nabla U = \bar{0}]$$

$$\Rightarrow -\iint(\nabla U\times\bar{V})\cdot\hat{n}\,ds = \iiint \nabla U\cdot\nabla\times\bar{V}\,dv$$

Q: If S is closed surface enclosing volume V, show that $\iiint_V \nabla\phi\cdot\operatorname{Curl}\bar{F}\,dV = \iint_S(\bar{F}\times\nabla\phi)\cdot\overline{ds}$

Sol^n: $\iint_S(\bar{F}\times\nabla\phi)\cdot\overline{ds} = \iint_S(\bar{F}\times\nabla\phi)\cdot\hat{n}\,ds$

$$= \iiint_V \operatorname{div}(\bar{F}\times\nabla\phi)\,dv \quad [\text{By divergence th}^m]$$

$$= \iiint_V[\nabla\phi\cdot\operatorname{Curl}\bar{F} - \bar{F}\cdot\operatorname{Curl}(\nabla\phi)]\,dv$$

$$= \iiint_V \nabla\phi\cdot\operatorname{Curl}\bar{F}\,dv \quad [\because \operatorname{Curl}(\nabla\phi) = \bar{0}]$$

Q: Prove that $\iiint_V \frac{dv}{r^2} = \iint_S \frac{\bar{r}\cdot\hat{n}}{r^2}\,ds$, where S is closed surface and V be volume bdded by S.

Sol^n: By divergence th^m, we have

$$\iint_S\left(\frac{\bar{r}}{r^2}\right)\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\left(\frac{1}{r^2}\bar{r}\right)dv$$

$$= \iiint_V\left[\frac{1}{r^2}(\nabla\cdot\bar{r}) + \left(\nabla\frac{1}{r^2}\right)\cdot\bar{r}\right]dv$$

$$= \iiint_V\left(\frac{3}{r^2} - \frac{2}{r^4}\bar{r}\cdot\bar{r}\right)dv \quad \left[\begin{array}{l}\because \nabla\cdot\bar{r} = 3\\ \text{and } \nabla r^n = n r^{n-2}\bar{r}\end{array}\right]$$

$$= \iiint_V\left(\frac{3}{r^2} - \frac{2}{r^2}\right)dv = \iiint_V \frac{dv}{r^2}$$

Q: If ϕ is harmonic in v, then $\iint_S \frac{\partial \phi}{\partial n} ds = 0$, where S is surface enclosing v.

Sol^n: $\iint_S \frac{\partial \phi}{\partial n} ds = \iint_S \left(\frac{\partial \phi}{\partial n}\vec{n}\right)\cdot\vec{n}\, ds = \iint_S \nabla\phi\cdot\vec{n}\, ds$

$= \iiint_V \operatorname{div}(\nabla\phi)\, dv$ (by divergence th^m) $\left[\because \nabla\phi = \frac{\partial\phi}{\partial n}\vec{n}\right]$

$= \iiint_V \nabla\cdot(\nabla\phi)\, dv = \iiint_V \nabla^2\phi\, dv = 0$ $\left[\because \nabla^2\phi = 0 \text{ as } \phi \text{ is harmonic in } v\right]$

Q: Express $\iiint [(\text{grad } f)\cdot\vec{v} + f \operatorname{div}\vec{v}]\, dv$, as surface integral.

Sol^n: $\iiint [(\text{grad. } f)\cdot\vec{v} + f\operatorname{div}\vec{v}]\, dv = \iiint \operatorname{div}(f\vec{v})\, dv$ $\left[\because \operatorname{div}(u\vec{v}) = (\text{grad } u)\cdot\vec{v} + u(\operatorname{div}\vec{v})\right]$

$= \iint (f\vec{v})\cdot\hat{n}\, ds$ (by divergence th^m)

Q: If ϕ is harmonic in V; then $\iint \phi\frac{\partial\phi}{\partial n} ds = \iiint |\nabla\phi|^2 dv$

Sol^n: $\iint \phi\frac{\partial\phi}{\partial n} ds = \iint \left(\phi\frac{\partial\phi}{\partial n}\hat{n}\right)\cdot\hat{n}\, ds$

$= \iint (\phi\nabla\phi)\cdot\hat{n}\, ds$ $\left[\because \text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial n}\hat{n}\right]$

$= \iiint \operatorname{div}(\phi\nabla\phi)\, dv$ (by divergence th^m)

$= \iiint [\nabla\phi\cdot\nabla\phi + \phi\, \nabla\cdot(\nabla\phi)]\, dv$ $\left[\because \operatorname{div}(u\vec{v}) = \nabla u\cdot\vec{v} + u(\nabla\cdot\vec{v})\right]$

$= \iiint [(\nabla\phi)^2 + \phi\nabla^2\phi]\, dv$

$= \iiint |\nabla\phi|^2 dv$ $\left[\because \nabla^2\phi = 0 \text{ as } \phi \text{ is harmonic in } V \text{ and } (\nabla\phi)^2 = |\nabla\phi|^2\right]$

Q: Show that $\iint \hat{n}\, ds = 0$ for any closed surface S.

Sol^n $\iint \hat{n}\, ds = \iint 1\,\hat{n}\, ds$ $\left[\because \iint_S \phi\hat{n}\, ds = \iiint_V \nabla\phi\, dv \text{ by th}^m \text{ on page } (4)\right]$

$= \iiint (\nabla 1)\, dv$

$= \vec{0}$ $[\because \nabla 1 = \vec{0}]$

Q: If S is any closed surface enclosing volume V. (12)
and $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, then show that $\iint_S \vec{F}\cdot\hat{n}\,ds = 6V$

Sol^n: $\operatorname{div}\vec{F} = 1+2+3 = 6$

By divergence th^m $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv = \iiint_V 6\,dv$

$= 6\iiint_V dv = 6V$

Cartesian Form of Divergence Theorem

Let $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be vector pt. function in component form, then $\operatorname{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$. — (1)

Let α, β, γ be angles which outward drawn unit vectors normal $\hat{n}$ makes with positive direction of axes.

Then $\langle \cos\alpha, \cos\beta, \cos\gamma \rangle$ are direction cosine of $\hat{n}$

$\therefore \hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$

and $\vec{F}\cdot\hat{n} = f_1\cos\alpha + f_2\cos\beta + f_3\cos\gamma$. — (2)

$\therefore$ Divergence th^m can be written as

$$\iiint_V \operatorname{div}\vec{F}\,dv = \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)dx\,dy\,dz \quad \text{(by (1))}$$

$$= \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S (f_1\cos\alpha + f_2\cos\beta + f_3\cos\gamma)\,ds \quad \text{(by (2))}$$

$$= \iint_S (f_1\,dy\,dz + f_2\,dz\,dx + f_3\,dx\,dy) \quad \left[\begin{array}{l} \because \cos\alpha\,ds = dy\,dz \\ \cos\beta\,ds = dz\,dx \\ \cos\gamma\,ds = dx\,dy \end{array}\right]$$

Q: (i) Evaluate $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$, where S is surface of sphere $x^2+y^2+z^2=a^2$

(ii) Evaluate $\iint_S [(x+z)\,dy\,dz + (y+z)\,dz\,dx + (x+y)\,dx\,dy]$, where S is surface of sphere $x^2+y^2+z^2=a^2$

~~(iii) Evaluate~~ $\cancel{\iint_S}$

Sol^n: (i) Using divergence th^m in Cartesian form, we have

$$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy) = \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dx\,dy\,dz$$

$$= \iiint_V (1+1+1)\,dx\,dy\,dz = 3\iiint_V dx\,dy\,dz = 3V$$

$$= 3\times(\text{volume of sphere } x^2+y^2+z^2=a^2) = 3\times\frac{4}{3}\pi a^3 = 4\pi a^3$$

(ii) Using divergence th^m in Cartesian form, we have

$$\iint_S [(x+z)\,dy\,dz + (y+z)\,dz\,dx + (x+y)\,dx\,dy]$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(x+y)\right] dx\,dy\,dz$$

$$= \iiint_V (1+1+0)\,dx\,dy\,dz = 2\times(\text{volume of sphere } x^2+y^2+z^2=a^2)$$

$$= 2\times\frac{4}{3}\pi a^3 = \frac{8}{3}\pi a^3$$

Q: Evaluate $\iint_S [x^2\,dy\,dz + y^2\,dz\,dx + 2z(xy-x-y)\,dx\,dy]$, where S is surface of cube $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$.

Sol^n: Using divergence th^m in Cartesian form, we have.

$$\iint_S [x^2\,dy\,dz + y^2\,dz\,dx + 2z(xy-x-y)\,dx\,dy]$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}\{2z(xy-x-y)\}\right] dV$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} (2x+2y+2xy-2x-2y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 2xy\,dx\,dy\,dz = \int_0^1\int_0^1 [x^2y]_0^1\,dy\,dz = \int_0^1\int_0^1 y\,dy\,dz$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dz = \frac{1}{2}[z]_0^1 = \frac{1}{2}(1-0) = \frac{1}{2}$$

Q: Evaluate $\iint_S (x^3\,dy\,dz + y^3\,dz\,dx + z^3\,dx\,dy)$, where S is surface of sphere $x^2+y^2+z^2=1$.

Sol^n: Using divergence th^m in Cartesian form.

$$\iint_S (x^3 dy dz + y^3 dz dx + z^3 dx dy) = \iiint_V \left[\frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)\right] dv$$

$$= \iiint_V (3x^2 + 3y^2 + 3z^2)\, dx\, dy\, dz$$

Using spherical polar co-ordinates
$x = r\cos\phi \sin\theta$, $y = r\sin\phi \sin\theta$
$z = r\cos\theta$
$\therefore x^2+y^2+z^2 = r^2$
and $0 \le r \le 1$, $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi$
$dv = r^2 \sin\theta \, dr\, d\theta\, d\phi$

$$= 3\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= 3\left[\frac{r^5}{5}\right]_0^1 \times \left[-\cos\theta\right]_0^{\pi} \times \left[\phi\right]_0^{2\pi}$$

$$= 3 \times \frac{1}{5} \times 2 \times 2\pi = \frac{12}{5}\pi$$

Q: Evaluate $\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$, where $\bar{F} = (x-z)\hat{i} + (x^3 + yz)\hat{j} - 3xy^2 \hat{k}$ and S is surface of $z = 2 - \sqrt{x^2+y^2}$ above the xy-plane

Sol^n: Here S is not closed surface.

The surface $z = 2 - \sqrt{x^2+y^2}$ meets xy plane in circle $x^2+y^2 = 4$, $z=0$. let S_1 be plane region bounded by circle. If S' is surface consists of surfaces S and S_1, then S' is closed surface enclosing volume V.

Then by divergence th^m, we have

$$\iint_{S'} (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \iiint_V \nabla \cdot (\nabla \times \bar{F})\, dv = 0 \qquad [\because \nabla \cdot (\nabla \times \bar{F}) = 0]$$

$$\Rightarrow \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds + \iint_{S_1} (\nabla \times \bar{F}) \cdot \hat{n}\, ds = 0 \qquad [\because S' \text{ consists of } S \text{ and } S_1]$$

$$\Rightarrow \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = -\iint_{S_1} (\nabla \times \bar{F}) \cdot \hat{n}\, ds = -\iint_{S_1} (\nabla \times \bar{F}) \cdot (-\hat{k})\, ds$$

$$= \iint_{S_1} (\nabla \times \bar{F}) \cdot \hat{k}\, ds \quad \text{—— (1)} \qquad [\because \text{on } S_1,\ \hat{n} = -\hat{k}]$$

Now
$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-z & x^3+yz & -3xy^2 \end{vmatrix} = (-6xy - y)\hat{i} + (-1+3y^2)\hat{j} + 3x^2 \hat{k}$$

$\therefore$ by (1), we get

$$\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \iint_{S_1} \left[(-6xy - y)\hat{i} + (-1+3y^2)\hat{j} + 3x^2 \hat{k}\right] \cdot \hat{k}\, ds$$

$$= \iint_{S_1} 3x^2\, ds = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} 3r^2\cos^2\theta \cdot r\,dr\,d\theta$$

$\because S_1 : x^2+y^2=4,\ z=0$ changing into polar coordinate $x = r\cos\theta,\ y = r\sin\theta$, then $0 \le r \le 2,\ 0 \le \theta \le 2\pi$ and $ds = r\,dr\,d\theta$

$$= 3\left[\frac{r^4}{4}\right]_0^2 \cdot \int_0^{2\pi} \cos^2\theta\, d\theta$$

$$= 12 \times 4 \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = 48 \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= 12\pi$$

Q: Using Gauss' divergence thm, evaluate $\iint_S (ax^2+by^2+cz^2)\, ds$ over sphere $x^2+y^2+z^2=1$.

OR

Show that $\iint_S (ax\hat{i}+by\hat{j}+cz\hat{k})\cdot\hat{n}\, ds = \frac{4\pi(a+b+c)}{3}$, where S is surface of sphere $x^2+y^2+z^2=1$.

Solⁿ: Let us first put the integral $\iint_S (ax^2+by^2+cz^2)\, ds$ in the form $\iint_S \vec{F}\cdot\hat{n}\, ds$

The vector normal to $\phi(x,y,z) = x^2+y^2+z^2-1=0$ is

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k} = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = x\hat{i}+y\hat{j}+z\hat{k} \quad \left[\because x^2+y^2+z^2=1\right]$$

Now we have to choose $\vec{F}$ such that

$$\vec{F}\cdot\hat{n} = ax^2+by^2+cz^2$$

$$\Rightarrow \vec{F}\cdot(x\hat{i}+y\hat{j}+z\hat{k}) = (ax\hat{i}+by\hat{j}+cz\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})$$

$$\Rightarrow \vec{F} = ax\hat{i}+by\hat{j}+cz\hat{k}$$

$$\therefore \iint_S (ax^2+by^2+cz^2)\, ds = \iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$
(By divergence thm)

$$= \iiint_V \nabla\cdot(ax\hat{i}+by\hat{j}+cz\hat{k})\, dv = \iiint_V (a+b+c)\, dv$$

$$= (a+b+c) \times (\text{volume of sphere } x^2+y^2+z^2=1) = (a+b+c)\frac{4}{3}\pi$$

Q: Verify Gauss divergence thm for function $\vec{F} = y\hat{i}+x\hat{j}+z^2\hat{k}$ over the cylinderical region bdded by $x^2+y^2=9$, $z=0$ and $z=2$.

Solⁿ: Let S denotes the entire surface of given cylinder and let V be volume enclosed by S.

We are to prove that $\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$ ——(1)

Now $\iiint_V \nabla\cdot\bar{F}\,dv = \iiint_V 2z\,dv = \int_{z=0}^{2}\int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 2z\,dy\,dx\,dz$

$= 2\int_0^2\int_{-3}^{3} z\left[y\right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx\,dz = 2\int_0^2\int_{-3}^{3} z\left(2\sqrt{9-x^2}\right)dx\,dz.$

$= 4\times 2\int_0^2\int_0^3 z\sqrt{9-x^2}\,dx\,dz = 8\left[\frac{z^2}{2}\right]_0^2 \times \left[\frac{x\sqrt{9-x^2}}{2}+\frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3$

$= 8\,(2-0)\left(0+\frac{9}{2}\times\frac{\pi}{2}-0\right) = 36\pi$

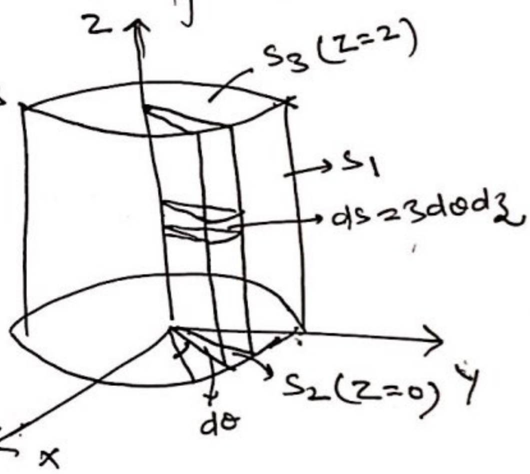
Now we calculate $\iint_S \bar{F}\cdot\hat{n}\,ds$ over surface S

Divide surface S into three surfaces

(i) S_1: curved surface of cylinder

(ii) S_2: surface of base of cylinder

(iii) S_3: surface of top of cylinder



$\therefore \iint_S \bar{F}\cdot\hat{n}\,ds = \iint_{S_1}\bar{F}\cdot\hat{n}\,ds + \iint_{S_2}\bar{F}\cdot\hat{n}\,ds + \iint_{S_3}\bar{F}\cdot\hat{n}\,ds$ ——(2)

For surface S_1: Let $\phi(x,y,z) = x^2+y^2-9=0$

$\therefore \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k} = 2x\hat{i}+2y\hat{j}$

and $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{x\hat{i}+y\hat{j}}{3}$ $[\because x^2+y^2=9]$

$\therefore \iint_{S_1}\bar{F}\cdot\hat{n}\,ds = \iint_{S_1}(y\hat{i}+x\hat{j}+z^2\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}}{3}\right)ds = \frac{2}{3}\iint_S xy\,ds$

The parametric eqⁿ of curved surface S_1 are $x=3\cos\theta$, $y=3\sin\theta$ and z being arbitrary

$\therefore ds =$ elementary area on the surface $S_1 = 3\,d\theta\,dz$.

Now for surface S_1, θ varies from 0 to 2π and z varies from 0 to 2

$$\therefore \iint_{S_1} \bar{F}\cdot\hat{n}\,ds = \frac{2}{3}\int_{z=0}^{2}\int_{\theta=0}^{2\pi}(3\cos\theta)(3\sin\theta)\,3\,d\theta\,dz$$

$$= 18\int_{\theta=0}^{2\pi}\int_{z=0}^{2}\sin\theta\cos\theta\,dz\,d\theta = 18\int_{\theta=0}^{2\pi}\sin 2\theta\,d\theta = 18\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= 9\left[\cos 0 - \cos 2\pi\right] = 9(1-1) = 0.$$

For surface S_2 $\quad z=0,\ \hat{n}=-\hat{k}$

$$\therefore \iint_{S_2}\bar{F}\cdot\hat{n}\,ds = \iint_{S_2}(y\hat{i}+x\hat{j}+0\hat{k})\cdot(-\hat{k})\,ds = \iint_{S_2} 0\,ds = 0$$

For surface S_3 $\quad z=2,\ \hat{n}=\hat{k}$

$$\therefore \iint_{S_3}\bar{F}\cdot\hat{n}\,ds = \iint_{S_3}(y\hat{i}+x\hat{j}+4\hat{k})\cdot\hat{k}\,ds = \iint_{S_3} 4\,ds$$

$$= 4S_3 = 4\times(\text{Area of circular top of radius } 3)$$

$$= 4\times\pi.(3)^2 = 36\pi$$

Thus $\iint_S \bar{F}\cdot\hat{n}\,ds = 0+0+36\pi = 36\pi$ [using (2)]

Thus $\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\bar{F}\,dv$

Hence Gauss' Divergence thm is verified.

Q: Evaluate $\iint_S \bar{F}\cdot\hat{n}\,ds$ over the entire region above the plane bdded by cone $z^2 = x^2+y^2$ and the plane $z=4$ if $\bar{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$.

Solⁿ: Here $\bar{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$

$$\therefore \operatorname{div}\bar{F} = 4z + xz^2 + 3$$

By divergence thm

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\bar{F}\,dv = \iiint_V (4z+xz^2+3)\,dx\,dy\,dz$$

where V is volume enclosed by surface S.

$$\therefore V = \{(x,y,z) : 0\le z\le 4,\ -z\le y\le z,\ -\sqrt{z^2-y^2}\le x\le\sqrt{z^2-y^2}\}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \int_{z=0}^{4}\int_{y=-z}^{z}\int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} (4z + xz^2 + 3)\,dx\,dy\,dz$$

$$= \int_0^4\int_{-z}^{z}\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} (4z+3)\,dx\,dy\,dz \qquad \left[\because xz^2 \text{ is odd function of } x\right]$$

$$= 2\int_0^4\int_{-z}^{z}\int_0^{\sqrt{z^2-y^2}} (4z+3)\,dx\,dy\,dz = 2\int_0^4\int_{-z}^{z} (4z+3)\sqrt{z^2-y^2}\,dy\,dz$$

$$= 4\int_0^4\int_0^z (4z+3)\sqrt{z^2-y^2}\,dy\,dz \qquad \left[\because (4z+3)\sqrt{z^2-y^2} \text{ is even function of } y\right]$$

$$= 4\int_0^4 \left[(4z+3)\left\{\frac{y}{2}\sqrt{z^2-y^2} + \frac{z^2}{2}\sin^{-1}\frac{y}{z}\right\}\right]_0^z dz$$

$$= 4\int_0^4 (4z+3)\left[\left(0+\frac{z^2}{2}\sin^{-1}1\right)-(0+0)\right]$$

$$= 4\int_0^4 (4z+3)\left(\frac{z^2}{2}\times\frac{\pi}{2}\right)dz = \pi\int_0^4 (4z^3+3z^2)\,dz$$

$$= \pi\left[z^4+z^3\right]_0^4 = \pi\left[(256+64)-(0+0)\right] = 320\pi$$